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Abstract. In 1949 Chowla and Selberg gave a very useful formula for the Epstein zeta-function associated with a positive definite binary quadratic form. Several generalizations of this formula are given here. The method of proof is new and is based on a theorem that we formerly proved for "generalized" Dirichlet series. An easy proof of Kronecker's second limit formula is also given.

1. Let $Q(m, n) = am^2+bmn+cn^2$ denote a positive definite quadratic form with a, b and c real. The Epstein zeta-function $Z(s, Q)$ is defined for $\operatorname{Re} s > 1$ by

$$Z(s, Q) = \sum_{m,n=-\infty}^{\infty}{}' \{Q(m, n)\}^{-s},$$

where the $'$ indicates that the term with $m = n = 0$ is omitted from the summation. In 1949 Chowla and Selberg [6] announced the following formula, valid in the entire complex plane, for $Z(s, Q)$:

$$\begin{aligned} a^s\Gamma(s)Z(s, Q) &= 2\Gamma(s)\zeta(2s)+2k^{1-2s}\pi^{1/2}\Gamma(s-\tfrac{1}{2})\zeta(2s-1) \\ &\quad + 8k^{1/2-s}\pi^s \sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n) \cos(n\pi b/a) K_{s-1/2}(2\pi kn). \end{aligned} \tag{1.1}$$

Here, $k^2 = |d|/4a^2$, where $d = b^2-4ac$, $\sigma_\nu(n) = \sum_{d|n} d^\nu$, ζ denotes the Riemann zeta-function, and K_ν denotes the modified Bessel function. Proofs of (1.1) were later given by Rankin [11], Bateman and Grosswald [3], Chowla and Selberg [7], and Motohashi [10]. The objective of this paper is to give several generalizations of (1.1).

Before proceeding further, we introduce some notation. We write $s = \sigma + it$ with σ and t both real. We denote the set of rational integers by Z. The summation sign $\sum$ with no indices will always mean $\sum_{n=1}^{\infty}$. If a summation sign appears with a $'$ on it, then the terms (if any) which make the summand meaningless are omitted from the sum. Lastly, we write $\sum'_{n_1,\dots,n_m}$ for

$$\sum_{n_1=-\infty}^{\infty}{}' \cdots \sum_{n_m=-\infty}^{\infty}{}' .$$

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First, we consider the more general Epstein zeta-function [8] defined for $\sigma>1$ by

$$Z(s, Q; g, h) = \sum_{m,n}{}' \exp(2\pi i\{h_1 m+h_2 n\})\{Q(m+g_1, n+g_2)\}^{-s}.$$

Here, $g=(g_1, g_2)$ and $h=(h_1, h_2)$, where g_1, g_2, h_1 and h_2 are real. In §2 we derive a formula for $Z(s, Q; g, h)$ for which (1.1) is a special case when $g=h=0=(0, 0)$.

In [3] and [7] it was shown how the functional equation for $Z(s, Q)$ can be derived from (1.1). Likewise, we show that the functional equation for $Z(s, Q; g, h)$ can be derived from our generalization of (1.1).

Chowla and Selberg [7] used (1.1) to give a very simple proof of Kronecker's first limit formula. Motohashi [10] has given a similar proof. In §4 we use our formula to give an easy proof of Kronecker's second limit formula.

In §5 we give a further generalization by deriving an analogue of (1.1) for

$$Z(s, Q; g, h) = \sum_{n_1,\ldots,n_m}{}' \exp\left(2\pi i \sum_{j=1}^{m} h_j n_j\right)\{Q(n_1+g_1, \ldots, n_m+g_m)\}^{-s},$$

where $\sigma>m/2$. Here, $Q(n_1, \ldots, n_m)$ is a positive definite quadratic form with real coefficients in m variables, and $g=(g_1, \ldots, g_m)$ and $h=(h_1, \ldots, h_m)$ are real vectors.

In §6 we derive a formula for

$$Z(s, Q, c) = \sum_{m,n}{}' c(n)\{Q(m, n)\}^{-s},$$

where σ is sufficiently large, and $c(n)$ is an arithmetical function subject to some fairly general conditions. In fact, it will be clear that a similar generalization of $Z(s, Q; g, h)$ follows in a similar way.

Lastly, we derive a formula for expressions of the form

$$\sum_{m,n=1}^{\infty} a^*(m)a(n)(\lambda_m^*+\lambda_n)^{-s},$$

where σ is sufficiently large, and $a^*(m)$, $a(n)$, λ_m^* and λ_n will be specified later. Barrucand [2] has derived a special case of our formula. It should be noted that Stark's formula [13] for

$$\sum_{m,n}{}' \chi(Q(m, n))\{Q(m, n)\}^{-s},$$

where $\sigma>1$, χ is a character, and Q has integral coefficients, is not covered by our results in §6 and §7.

The proofs of (1.1) by Rankin [11], Bateman and Grosswald [3], and Chowla and Selberg [7] use the theory of Fourier series. The proof of Motohashi [10] uses Poisson's summation formula. Chowla and Selberg [7] give another proof of (1.1) based on the functional equation of theta-series. Our generalizations in sections 2, 5, 6 and 7 can be established by this latter method. However, we shall give perhaps even easier proofs based on a theorem that we first established in [4] for "generalized" Dirichlet series. A simpler proof, as well as a better statement of the theorem, is given in [5]. To state that theorem we shall need some definitions.

DEFINITION 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers strictly increasing to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Consider the functions φ and ψ representable as Dirichlet series

$$\varphi(s) = \sum a(n)\lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum b(n)\mu_n^{-s}$$

with finite abscissas of absolute convergence σ_a and σ_a', respectively. If r is real, we say that φ and ψ satisfy the functional equation

$$\Gamma(s)\varphi(s) = \Gamma(r-s)\psi(r-s)$$

if there exists a meromorphic function χ with the properties:

(i) $\chi(s)=\Gamma(s)\varphi(s)$, $\sigma>\sigma_a$, $\chi(s)=\Gamma(r-s)\psi(r-s)$, $\sigma<r-\sigma_a'$;

(ii) $\lim_{|t|\to\infty} \chi(\sigma+it)=0$, uniformly in every interval $-\infty<\sigma_1\leqq\sigma\leqq\sigma_2<\infty$;

(iii) the poles of χ are confined to some compact set.

DEFINITION 2. Let φ satisfy Definition 1. For $x>0$ and $\sigma>\sigma_a$ we define the generalized Dirichlet series $\varphi(s, x)$ by

$$\varphi(s, x) = \sum a(n)(x+\lambda_n)^{-s}.$$

THEOREM 1 [4], [5]. *Let $\mathscr{D}$ be a domain where*

$$\sum b(n)\mu_n^{(s-r)/2}K_{s-r}(2(\mu_n x)^{1/2})$$

converges uniformly. Let $R(s, x)$ denote the sum of the residues of $\chi(w)\Gamma(s-w)x^{w-s}$ at the poles of $\chi(w)$. Then, if $s\in\mathscr{D}$,

$$\Gamma(s)\varphi(s, x) = R(s, x)+2\sum b(n)(\mu_n/x)^{(s-r)/2}K_{s-r}(2(\mu_n x)^{1/2}). \tag{1.2}$$

2. THEOREM 2. *Let $y=b/2a$ and $k^2=|d|/4a^2$. Define*

$$f_1(s) = k^s\pi^{-s}\Gamma(s)\exp(-2\pi i g_2h_2)l_1(s), \qquad g_2\in Z,$$
$$= 0, \qquad g_2\notin Z,$$

and

$$f_2(s) = k^{1-s}\pi^{1/2-s}\Gamma(s-\tfrac{1}{2})l_2(s-\tfrac{1}{2}), \qquad h_1\in Z,$$
$$= 0, \qquad h_1\notin Z,$$

where for $\sigma>\frac{1}{2}$

$$l_1(s) = \sum_m{}' \frac{\exp(2\pi i h_1 m)}{|m+g_1|^{2s}}$$

and

$$l_2(s) = \sum_n{}' \frac{\exp(2\pi i h_2 n)}{|n+g_2|^{2s}}.$$

Then for all s,

$$\begin{aligned}(2\pi/|d|^{1/2})^{-s}\Gamma(s)Z(s, Q; g, h) &= f_1(s)+f_2(s)+2k^{1/2}\sum_{m,n}{}' \exp(2\pi i\{-(h_1+m)[g_1+y(n+g_2)]+h_2n\}) \\ &\quad\cdot|(m+h_1)/(n+g_2)|^{s-1/2}K_{s-1/2}(2\pi k|m+h_1|\,|n+g_2|).\end{aligned} \tag{2.1}$$

Proof. The pair of functions

$$\varphi^*(s) = \sum_m{}' \frac{\exp(2\pi i h_1 m)}{|(m+g_1)+y(n+g_2)|^{2s}}$$

and

$$\psi^*(s) = \sum_m{}' \frac{\exp(-2\pi i m[g_1+y(n+g_2)])}{|m+h_1|^{2s}}$$

satisfy the functional equation [8, p. 207]

$$\pi^{-s}\Gamma(s)\varphi^*(s) = \pi^{s-1/2}\Gamma(\tfrac{1}{2}-s)\exp(-2\pi i h_1[g_1+y(n+g_2)])\psi^*(\tfrac{1}{2}-s).$$

We apply Theorem 1 to $\varphi(s)=\pi^{-s}\varphi^*(s)$ and

$$\psi(s) = \pi^{-s}\exp(-2\pi i h_1[g_1+y(n+g_2)])\psi^*(s).$$

We choose $x=\pi k^2(n+g_2)^2$, $n+g_2\neq 0$. Thus, for $\sigma>\frac{1}{2}$,

$$\begin{aligned}(2.2)\qquad &\pi^{-s}\Gamma(s)\sum_{\substack{m\\ m+g_1+y(n+g_2)\neq 0}} \frac{\exp(2\pi i h_1 m)}{\{k^2(n+g_2)^2+[(m+g_1)+y(n+g_2)]^2\}^s}\\ &= R(s,\pi k^2(n+g_2)^2)+2\sum_m{}'\exp(-2\pi i(h_1+m)[g_1+y(n+g_2)])\\ &\cdot|(m+h_1)/k(n+g_2)|^{s-1/2}K_{s-1/2}(2\pi k|m+h_1|\,|n+g_2|).\end{aligned}$$

Now, $\Gamma(w)\varphi^*(w)$ has possible poles at $w=0$ and $w=\frac{1}{2}$ [8, pp. 207–208]. If $\{g_1+y(n+g_2)\}\in Z$, $\Gamma(w)\varphi^*(w)$ has a simple pole at $w=0$ with residue $-\exp(-2\pi i h_1[g_1+y(n+g_2)])$. Otherwise, $\Gamma(w)\varphi^*(w)$ is analytic at $w=0$. If $h_1\in Z$, $\Gamma(w)\varphi^*(w)$ has a simple pole at $w=\frac{1}{2}$ with residue $\pi^{1/2}$. Otherwise, $\Gamma(w)\varphi^*(w)$ is analytic at $w=\frac{1}{2}$. Thus, substituting the value of $R(s,\pi k^2(n+g_2)^2)$ into (2.2), multiplying both sides by $k^s\exp(2\pi i h_2 n)$, and then summing on n, $-\infty<n<\infty$, $n+g_2\neq 0$, we obtain for $\sigma>1$,

$$\begin{aligned}(2.3)\qquad &(\pi/k)^{-s}\Gamma(s)\sum_{\substack{m,n\\ m+g_1+y(n+g_2)\neq 0\\ n+g_2\neq 0}} \frac{\exp(2\pi i\{h_1 m+h_2 n\})}{\{k^2(n+g_2)^2+[(m+g_1)+y(n+g_2)]^2\}^s}\\ &= f_2(s)-f_3(s)+2k^{1/2}\sum_{m,n}{}'\exp(2\pi i\{-(h_1+m)[g_1+y(n+g_2)]+h_2 n\})\\ &\cdot|(m+h_1)/(n+g_2)|^{s-1/2}K_{s-1/2}(2\pi k|m+h_1|\,|n+g_2|),\end{aligned}$$

where

$$f_3(s) = (\pi k)^{-s}\Gamma(s)\sum_{\substack{n\\ g_1+y(n+g_2)\in Z}}{}' \frac{\exp(2\pi i\{h_2 n-h_1[g_1+y(n+g_2)]\})}{|n+g_2|^{2s}}.$$

We add $f_3(s)$ to both sides of (2.3) and note that $f_3(s)$ comprises exactly the "missing" terms of the series on the left when $\{g_1+y(n+g_2)\}\in Z$ and $n+g_2\neq 0$. Next, we add $f_1(s)$ to both sides of (2.3) and note that $f_1(s)$ comprises exactly the "missing" terms of the series on the left when $n+g_2=0$ but $m+g_1\neq 0$. The resulting series on the left is then $a^sZ(s, Q; g, h)$. We thus arrive at (2.1), and the theorem follows by analytic continuation.

3. We now derive from (2.1) the functional equation of $Z(s, Q; g, h)$, first given by Epstein [8].

THEOREM 3. *Let Q^{-1} denote the inverse of Q. Then,*

$$(2\pi/|d|^{1/2})^{-s}\Gamma(s)Z(s, Q; g, h) = \exp(-2\pi i\{g_1h_1+g_2h_2\})(2\pi/|d|^{1/2})^{s-1}\Gamma(1-s)Z(1-s, Q^{-1}; h, -g).$$

Proof. Suppose that we reprove Theorem 2 by summing on n first instead of m. Then in (2.1) $m, n, g_1, g_2, h_1, h_2, y=b/2a$ and $k^2=|d|/4a^2$ are replaced by $n, m, g_2, g_1, h_2, h_1, y'=b/2c$ and $k'^2=|d|/4c^2$, respectively. We thus obtain

$$(3.1)\quad \begin{aligned}(2\pi/|d|^{1/2})^{-s}\Gamma(s)Z(s, Q; g, h) &= f_4(s)+f_5(s)+2k'^{1/2}\sum_{m,n}{}' \exp(2\pi i\{-(h_2+n)[g_2+y'(m+g_1)]+h_1m\}) \\ &\quad \cdot|(n+h_2)/(m+g_1)|^{s-1/2}K_{s-1/2}(2\pi k'|n+h_2|\ |m+g_1|),\end{aligned}$$

where

$$\begin{aligned} f_4(s) &= k'^s\pi^{-s}\Gamma(s)\exp(-2\pi i g_1h_1)l_2(s), && g_1\in Z, \\ &= 0, && g_1\notin Z,\end{aligned}$$

and

$$\begin{aligned} f_5(s) &= k'^{1-s}\pi^{1/2-s}\Gamma(s-\tfrac{1}{2})l_1(s-\tfrac{1}{2}), && h_2\in Z, \\ &= 0, && h_2\notin Z.\end{aligned}$$

Next, we return again to (2.1) and replace Q, g and h by Q^{-1}, h and $-g$, respectively. Note that $Q^{-1}(m, n)=cm^2-bmn+an^2$. Thus, in (2.1) g_1, g_2, h_1, h_2, y and k^2 are replaced by $h_1, h_2, -g_1, -g_2, -y'$ and k'^2, respectively. Hence,

$$(3.2)\quad \begin{aligned}\exp(-2\pi i\{g_1h_1+g_2h_2\})(2\pi/|d|^{1/2})^{-s}\Gamma(s)Z(s, Q^{-1}; h, -g) &= f_6(s)+f_7(s) \\ +2k'^{1/2}\sum_{m,n}{}' \exp(2\pi i\{-g_2h_2-g_1y'(n+h_2)-m[h_1-y'(n+h_2)]-g_2n\}) & \\ \cdot|(m-g_1)/(n+h_2)|^{s-1/2}K_{s-1/2}(2\pi k'|m-g_1|\ |n+h_2|),&\end{aligned}$$

where

$$\begin{aligned} f_6(s) &= k'^s\pi^{-s}\Gamma(s)\exp(-2\pi i g_1h_1)l_3(s), && h_2\in Z, \\ &= 0, && h_2\notin Z,\end{aligned}$$

and

$$\begin{aligned} f_7(s) &= k'^{1-s}\pi^{1/2-s}\Gamma(s-\tfrac{1}{2})\exp(-2\pi i\{g_1h_1+g_2h_2\})l_4(s-\tfrac{1}{2}), && g_1\in Z, \\ &= 0, && g_1\notin Z,\end{aligned}$$

where for $\sigma>\frac{1}{2}$

$$l_3(s) = \sum_m{}' \frac{\exp(-2\pi i g_1 m)}{|m+h_1|^{2s}} \quad \text{and} \quad l_4(s) = \sum_n{}' \frac{\exp(-2\pi i g_2 n)}{|n+h_2|^{2s}}.$$

In (3.1) replace s by $1-s$ and m by $-m$ in the last sum on the right-hand side. Using the fact that $K_\nu(x)=K_{-\nu}(x)$, we obtain

$$\begin{aligned}(3.3)\quad &(2\pi/|d|^{1/2})^{s-1}\Gamma(1-s)Z(1-s, Q; g, h)\\ &= f_4(1-s)+f_5(1-s)+2k'^{1/2}\sum_{m,n}{}' \exp(2\pi i\{-(h_2+n)[g_2+y'(-m+g_1)]-h_1 m\})\\ &\qquad \cdot|(m-g_1)/(n+h_2)|^{s-1/2}K_{s-1/2}(2\pi k'|n+h_2|\,|m-g_1|).\end{aligned}$$

We note that the last series on the right-hand sides of (3.2) and (3.3) are the same. Now [8, p. 207],

$$\pi^{-s}\Gamma(s)l_1(s) = \exp(-2\pi i g_1 h_1)\pi^{s-1/2}\Gamma(\tfrac{1}{2}-s)l_3(\tfrac{1}{2}-s)$$

and

$$\pi^{-s}\Gamma(s)l_2(s) = \exp(-2\pi i g_2 h_2)\pi^{s-1/2}\Gamma(\tfrac{1}{2}-s)l_4(\tfrac{1}{2}-s).$$

From these two functional equations it readily follows that $f_6(s)=f_5(1-s)$ and $f_7(s)=f_4(1-s)$, respectively. Thus, (3.2) and (3.3) agree, and Theorem 3 follows.

4. We now give a simple proof of Kronecker's second limit formula. Our proof is apparently shorter than other known proofs, e.g. that in [12].

First recall the definitions of the classical eta and theta functions:

$$\eta(z) = e^{\pi i z/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n z}), \qquad \operatorname{Im} z > 0,$$

and, for w complex,

$$\begin{aligned}\theta_1(w, z) = &-ie^{\pi i z/4}(e^{\pi i w}-e^{-\pi i w})\\ &\cdot\prod_{n=1}^{\infty}(1-e^{2\pi i(w+nz)})(1-e^{-2\pi i(w-nz)})(1-e^{2\pi i n z}), \qquad \operatorname{Im} z > 0.\end{aligned}$$

THEOREM 4. *Let $z=y+ik$ and $w=h_2-h_1 z$, where $h_1\notin Z$. Then,*

$$(4.1)\qquad \frac{z-\bar z}{2i}\sum_{m,n}{}' \frac{\exp(2\pi i\{mh_1+nh_2\})}{|m+nz|^2} = -2\pi\log\left|\frac{\theta_1(w, z)}{\eta(z)}\right| - \frac{\pi^2 i(w-\bar w)^2}{z-\bar z}.$$

Proof. We use (2.1) with $g=0$. We have assumed that $h_1\notin Z$. We could have assumed that $h_2\notin Z$. We would then interchange h_1 and h_2 in the definition of w and use (3.1) instead of (2.1). We also assume without loss of generality that $0<h_1<1$. We see from (2.1), or from [8, p. 207], that $Z(s, Q; 0, h)$ is analytic at $s=1$. Putting $s=1$ in the last expression on the right-hand side of (2.1) and using the fact that $K_{1/2}(z)=(\pi/2z)^{1/2}e^{-z}$, we find that this series becomes

$$\sum_{m,n}' |n|^{-1} \exp(2\pi i\{-h_1yn-mny+nh_2+ik|m+h_1|\,|n|\})$$

$$= \left(\sum_{m=-\infty}^{-1}+\sum_{m=0}^{\infty}\right)\sum_{n=1}^{\infty} n^{-1}[\exp(2\pi in\{-h_1y-my+h_2+ik|m+h_1|\}) + \exp(2\pi in\{h_1y+my-h_2+ik|m+h_1|\})]$$

$$= -\log\prod_{m=1}^{\infty}[1-\exp(2\pi i\{-h_1y+my+h_2+ik(m-h_1)\})] \cdot [1-\exp(2\pi i\{h_1y-my-h_2+ik(m-h_1)\})]$$

$$-\log\prod_{m=0}^{\infty}[1-\exp(2\pi i\{-h_1y-my+h_2+ik(m+h_1)\})] \cdot [1-\exp(2\pi i\{h_1y+my-h_2+ik(m+h_1)\})]$$

$$= -\log\prod_{m=1}^{\infty}[1-e^{2\pi i(w+mz)}][1-e^{-2\pi i(\bar{w}+m\bar{z})}]$$

$$-\log\prod_{m=0}^{\infty}[1-e^{2\pi i(\bar{w}-m\bar{z})}][1-e^{-2\pi i(w-mz)}]$$

$$= -\log[\theta_1(w,z)(ie^{-\pi iz/4}\{e^{\pi iw}-e^{-\pi iw}\}^{-1})]$$
$$-\log[\overline{\theta_1(w,z)}(-ie^{\pi i\bar{z}/4}\{e^{-\pi i\bar{w}}-e^{\pi i\bar{w}}\}^{-1})]$$
$$+\log(\eta(z)e^{-\pi iz/12})+\log(\overline{\eta(z)}e^{\pi i\bar{z}/12})$$
$$-\log(1-e^{-2\pi iw})-\log(1-e^{2\pi i\bar{w}})$$
$$= -2\log|\theta_1(w,z)/\eta(z)|+\pi i(w-\bar{w})+\pi i(z-\bar{z})/6.$$

Thus, from Theorem 2 we have

$$(2\pi/|d|^{1/2})^{-1}Z(1,Q;0,h) = kl_2(1)/\pi-2\log|\theta_1(w,z)/\eta(z)|+\pi i(w-\bar{w})+\pi i(z-\bar{z})/6.$$

Rewriting $Z(1,Q;0,h)$ in another form [12, p. 6] and noting that $z-\bar{z}=i|d|^{1/2}/a$, we have

$$\frac{z-\bar{z}}{2i}\sum_{m,n}'\frac{\exp(2\pi i\{mh_1+nh_2\})}{|m+nz|^2} = kl_2(1)-2\pi\log|\theta_1(w,z)/\eta(z)|+\pi^2i(w-\bar{w})+\pi^2i(z-\bar{z})/6. \tag{4.2}$$

We now observe that [1, p. 805]

$$B_2(h_1) = h_1^2-h_1+\frac{1}{6} = \frac{1}{\pi^2}\sum\frac{\cos(2\pi mh_1)}{m^2} = \frac{1}{2\pi^2}l_2(1),$$

where $B_2(x)$ is the second Bernoulli polynomial. Substituting the above value of $l_2(1)$ into (4.2), using the fact that $h_1=(\bar{w}-w)/(z-\bar{z})$, and simplifying, we arrive at (4.1).

5. Let

$$Q(n_1,\ldots,n_m) = \sum_{i,j=1}^{m} a_{ij}n_in_j, \qquad a_{ij} = a_{ji},$$

be a positive definite quadratic form in m variables with a_{ij} real. Without loss of generality assume that $a_{11}\neq 0$. Then, we can write [9, p. 231]

$$Q(n_1,\ldots,n_m) = a_{11}\{(n_1+b_{12}n_2+\cdots+b_{1m}n_m)^2+Q_1(n_2,\ldots,n_m)\},$$

where $b_{1j}=a_{1j}/a_{11}$, $j=2,\ldots,m$, and $Q_1(n_2,\ldots,n_m)$ is a positive definite quadratic form in $m-1$ variables. If b_{ij}, $i,j=2,\ldots,m$, are the coefficients of Q_1, then $b_{ij}=a_{ij}/a_{11}-a_{1i}a_{1j}/a_{11}^2$. Using this representation of Q, we can now give an extension of Theorem 2 to arbitrary Epstein zeta-functions.

THEOREM 5. *Let*

$$\begin{aligned} f_8(s) &= \pi^{-s}\Gamma(s)\exp\left(-2\pi i\sum_{j=2}^m g_jh_j\right)l_1(s), && g_2,\ldots,g_m\in Z,\\ &= 0, && \textit{otherwise}, \end{aligned}$$

and

$$\begin{aligned} f_9(s) &= \pi^{1/2-s}\Gamma(s-\tfrac12)Z_1(s-\tfrac12, Q_1; g, h), && h_1\in Z,\\ &= 0, && h_1\notin Z, \end{aligned}$$

where $l_1(s)$ is given in Theorem 2 and $Z_1(s, Q; g, h)$ is the Epstein zeta-function given for $\sigma>(m-1)/2$ by

$$Z_1(s, Q_1; g, h) = \sum_{n_2,\ldots,n_m}{}' \exp\left(2\pi i\sum_{j=2}^m h_jn_j\right)\{Q_1(n_2+g_2,\ldots,n_m+g_m)\}^{-s}.$$

For brevity, let $Q_1=Q_1(n_2+g_2,\ldots,n_m+g_m)$. Then, for all s,

$$\begin{aligned} \pi^{-s}\Gamma(s)a_{11}^sZ(s, Q; g, h) &= f_8(s)+f_9(s)\\ +2\sum_{n_1,\ldots,n_m}{}' &\exp\left(2\pi i\left\{-(h_1+n_1)[g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)]+\sum_{j=2}^m h_jn_j\right\}\right)\\ &\cdot|(n_1+h_1)/Q_1|^{s-1/2}K_{s-1/2}(2\pi|n_1+h_1|\,|Q_1|). \end{aligned} \tag{5.1}$$

If $h_1\in Z$, we can apply the theorem again to $Z_1(s-\frac12, Q_1; g, h)$. It is clear that we can successively apply the theorem until we have expressed the left-hand side of (5.1) in terms of products of Γ-functions and zeta-functions involving only 1 variable of summation, and in terms of infinite series of Bessel functions. We can also develop analogues of Kronecker's limit formulas, although it does not seem possible in general to express the infinite series of Bessel functions in closed form.

Proof of Theorem 5. We consider the same pair of functions $\varphi^*(s)$ and $\psi^*(s)$ as in the proof of Theorem 2 except that $g_1+y(n+g_2)$ is replaced by $g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)$ and the variable of summation m in Theorem 2 is replaced by n_1. We apply Theorem 1 again with now $x=\pi Q_1$, $Q_1\neq 0$. Thus, for $\sigma>\frac12$,

$$\begin{aligned} \pi^{-s}\Gamma(s)\sum_{\substack{n_1\\ n_1+g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)\neq 0}} &\frac{\exp(2\pi ih_1n_1)}{\{Q_1+[(n_1+g_1)+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)]^2\}^s}\\ = R(s,\pi Q_1)+2\sum_{n_1}{}' &\exp(-2\pi i(h_1+n_1)[g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)])\\ &\cdot|(n_1+h_1)/Q_1|^{s-1/2}K_{s-1/2}(2\pi|n_1+h_1|\,|Q_1|). \end{aligned} \tag{5.2}$$

We proceed as in the proof of Theorem 2. If $\{g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)\} \in Z$, $\Gamma(w)\varphi^*(w)$ has a simple pole at $w=0$ with residue

$$-\exp(-2\pi i h_1[g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)]).$$

If $h_1 \in Z$, $\Gamma(w)\varphi^*(w)$ has a simple pole at $w=\frac{1}{2}$ with residue $\pi^{1/2}$. Otherwise, $\Gamma(w)\varphi^*(w)$ is analytic. We then substitute the values of $R(s, \pi Q_1)$ into (5.2), multiply both sides by $\exp(2\pi i \sum_{j=2}^{m} h_j n_j)$ and sum over $n_2, \ldots, n_m$, $-\infty < n_2, \ldots, n_m < \infty$, $Q_1 \neq 0$, with $\sigma > m/2$. We next observe that the resulting series on the right-hand side arising from the residue terms at $w=0$ is, except for a factor of -1, composed of precisely the "missing" terms on the left-hand side when $\{g_1+b_{12}(n_2+g_2)+\cdots+b_{1m}(n_m+g_m)\} \in Z$ and $Q_1 \neq 0$. We add this series to each side of the equation. The series on the right-hand side arising from the residue terms at $w=\frac{1}{2}$ is precisely $f_9(s)$. We then add $f_8(s)$ to both sides and note that Q_1 can achieve the value 0 if and only if $g_2, \ldots, g_m \in Z$. Thus, $f_8(s)$ consists of precisely the "missing" terms on the left-hand side when $Q_1=0$ but $n_1+g_1 \neq 0$. The resulting series on the left-hand side is then $a_{11}^s Z(s, Q; g, h)$, and we thus obtain (5.1). The theorem now follows by analytic continuation.

6. THEOREM 6. *Let $c(n)$ be an arithmetical function such that $c(n)=c(-n)$ and $c(n)=O(n^\alpha)$, $\alpha \geqq 0$, as n tends to ∞. For $\sigma > (\alpha+2)/2$, let*

$$Z(s, Q, c) = \sum_{m,n}{}' c(n)\{Q(m, n)\}^{-s},$$

and for $\sigma > \alpha+1$, let

$$C(s) = \sum c(n)n^{-s}.$$

Then, for $\sigma > (\alpha+2)/2$,

$$\pi^{-s}\Gamma(s)a^s Z(s, Q, c) = 2c(0)\pi^{-s}\Gamma(s)\zeta(2s)+2k^{1-2s}\pi^{1/2-s}\Gamma(s-\tfrac{1}{2})C(2s-1) + 8k^{1/2-s}\sum n^{s-1/2}\rho_{1-2s}(n)\cos(n\pi b/a)K_{s-1/2}(2\pi k n), \tag{6.1}$$

where $\rho_\nu(n)=\sum_{d \mid n} c(d)d^\nu$.

If $C(2s-1)$ has an analytic continuation to $\sigma > \sigma_0$, say, then (6.1) provides an analytic continuation of $Z(s, Q, c)$ to $\sigma > \sigma_0$. In particular, if $\varphi(s)=\lambda^{-s}C(s)$ satisfies Definition 1 for some constant $\lambda > 0$, then (6.1) holds for all s and provides an analytic continuation of $Z(s, Q, c)$ that is valid in the entire complex plane. If the singularities of $C(s)$ are known, then the singularities of $Z(s, Q, c)$ can be found since the last expression on the right-hand side of (6.1) represents an entire function.

The condition that $c(n)$ be even is not strictly necessary, but then the form of (6.1) would be altered somewhat. It will be clear from the proof below that one can state a similar theorem for a corresponding generalization of $Z(s, Q; g, h)$ when Q is binary. And in fact, it will also be clear how to extend Theorem 6 even further to generalizations of the Epstein zeta-functions of Theorem 5 with $c(n)$

replaced by a product of $m-1$ arithmetical functions. Also, analogues of Kronecker's limit formulas can be established.

Proof of Theorem 6. First, it is easy to see that $Z(s, Q, c)$ and $C(2s-1)$ do, in fact, converge absolutely for $\sigma>(\alpha+2)/2$.

The functions φ and ψ defined for $\sigma>\frac{1}{2}$ by

$$\varphi(s) = \pi^{-s}\sum_{m}{}' |m+yn|^{-2s} \quad\text{and}\quad \psi(s) = 2\pi^{-s}\sum \frac{\cos(2\pi ymn)}{m^{2s}}$$

satisfy the functional equation [8, p. 207] $\Gamma(s)\varphi(s)=\Gamma(\frac{1}{2}-s)\psi(\frac{1}{2}-s)$. We apply Theorem 1 with $x=\pi k^2n^2$, $n\neq 0$, and obtain, for $\sigma>\frac{1}{2}$,

$$\begin{aligned}\pi^{-s}\Gamma(s) \sum_{\substack{m\\ m+yn\neq 0}} &\{k^2n^2+(m+yn)^2\}^{-s}\\ &= -\Gamma(s)(\pi k^2n^2)^{-s}\delta(yn)+\Gamma(s-\tfrac{1}{2})(\pi k^2n^2)^{1/2-s}\\ &\quad +4k^{1/2-s}\sum \cos(2\pi mny)|m/n|^{s-1/2}K_{s-1/2}(2\pi km|n|),\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}\delta(yn) &= 1, \qquad yn\in Z,\\ &= 0, \qquad yn\notin Z.\end{aligned}$$

Now multiply both sides of (6.2) by $c(n)$ and sum over n, $-\infty<n<\infty$, $n\neq 0$, for $\sigma>(\alpha+2)/2$. Upon transferring the resulting first series on the right-hand side to the left-hand side and using the fact that $c(n)$ is even, we have for $\sigma>(\alpha+2)/2$

$$\begin{aligned}\pi^{-s}\Gamma(s) \sum_{\substack{m,n\\ n\neq 0}} &c(n)\{k^2n^2+(m+yn)^2\}^{-s}\\ &= 2k^{1-s}\pi^{1/2-s}\Gamma(s-\tfrac{1}{2})C(2s-1)\\ &\quad +8k^{1/2-s}\sum n^{s-1/2}\rho_{1-2s}(n)\cos(n\pi b/a)K_{s-1/2}(2\pi kn).\end{aligned}\tag{6.3}$$

Now add to both sides of (6.3) the "missing" terms corresponding to $n=0$, $m\neq 0$, on the left. We arrive at (6.1).

7. Theorem 7. *Let $\{\lambda_n^*\}$ be a positive sequence strictly increasing to ∞. Let $\{a^*(n)\}$ be a sequence of complex numbers such that $\varphi^*(s)=\sum a^*(n)\lambda_n^{*-s}$ has a finite abscissa of absolute convergence, say σ_a^*. Let $\varphi(s)=\sum a(n)\lambda_n^{-s}$ and $\psi(s)=\sum b(n)\mu_n^{-s}$ satisfy Definition* 1. *Let $\{\nu_j\}=\{\lambda_m^*\mu_n\}$, $m, n, j=1, 2, \ldots$, arranged in increasing order, and*

$$c_\kappa(j) = \sum_{\lambda_m^*\mu_n=\nu_j} a^*(m)b(n)\lambda_m^{*\kappa}.$$

Let $\mathscr{D}^*\subseteq\mathscr{D}$, where $\mathscr{D}$ is given in Theorem 1, *be a domain where*

$$F(s) = 2\sum c_{r-s}(j)\nu_j^{(s-r)/2}K_{s-r}(2\nu_j^{1/2})$$

converges uniformly. Then, for $s\in\mathscr{D}^*$ and $\sigma>\sigma_a^*+\sup(\sigma_a, \sigma_a^*)$,

$$\Gamma(s)\sum_{m,n=1}^{\infty} a^*(m)a(n)(\lambda_m^*+\lambda_n)^{-s} = \sum a^*(m)R(s, \lambda_m^*)+F(s).\tag{7.1}$$

Comments similar to those that we made after Theorem 6 about analytic continuation can be made here as well.

Proof of Theorem 7. The proof is simple. Consider (1.2) with $x=\lambda_m^*$. Multiply both sides of (1.2) by $a^*(m)$ and sum over m, $1 \leqq m < \infty$. We arrive at (7.1) after a slight simplification. It is easily seen that the series on the left side of (7.1) converges absolutely for $\sigma > \sigma_a + \sigma_a^*$. Since $\varphi^*(s)$ is analytic for $\sigma > \sigma_a^*$, we see from the definition of $R(s, x)$ that $\sum a^*(m)R(s, \lambda_m^*)$ converges absolutely for $\sigma > 2\sigma_a^*$. This concludes the proof.

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